On Designs of Maximal (+1, −1)-Matrices of Order $n \equiv 2 \pmod 4$. II*

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Abstract. Finding maximal $(+1, -1)$-matrices M_{2m} of order $2m$ (with odd m) constructible in the standard form

$$\begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}$$

is reduced to the finding of two polynomials $C(w)$, $D(w)$ (corresponding to the circulant submatrices A, B) satisfying

$$(*) \qquad |C(w)|^2 + |D(w)|^2 = \tfrac{1}{2}(m-1),$$

where w is any primitive mth root of unity. Thus, all M_{2m} constructible by the standard form (see [4]) can be classified by the formula (*). Some new matrices M_{2m} for $m = 25, 27, 31$, were found by this method. ■

Let M_{2m} be a maximal $(+1, -1)$-matrix of order $2m$ and let $S = ((s_i))$ be the circulant matrix of order m with the first row entries s_i $(0 \leqq i \leqq m-1)$, all zero but $s_1 = 1$.

When m is odd, it is known that (for $m \leqq 27$, except $m = 11, 17$; see [1]−[4]), M_{2m} can be constructed by the following matrix:**

$$(1) \qquad R = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}, \qquad \text{where } A = \sum_{k=0}^{m-1} a_k S^k,\ B = \sum_{k=0}^{m-1} b_k S^k \text{ with}$$

a_k and b_k, 1 or -1, and T indicates the transposed matrix. Then the gramian matrix of R becomes

$$RR^T = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},$$

where P is equal to

$$(2) \qquad AA^T + BB^T = 2\left(mI + \sum_{k=1}^{m-1} S^k\right),$$

where I is the identity matrix of order m.

By applying to the both sides of (2) the transformation L which transforms S into a diagonal matrix $W = [w_1, \cdots, w_m]$ with w_j, all distinct mth roots of unity, (namely, $L(S) = U^*SU = W$, where U is unitary and * indicates the conjugate transpose; see [5]) we obtain

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** For $n = 25$, without circulancy of submatrices A and B, see [2]; also see Addition of this paper.

$$A(w)A(w^{m-1}) + B(w)B(w^{m-1}) = 2\left(m + \sum_{k=1}^{m-1} w^k\right) \tag{3}$$

where

$$A(w) = \sum_{k=0}^{m-1} a_k w^k,\ B(w) = \sum_{k=0}^{m-1} b_k w^k,$$

and w is any mth root of unity.

Since w and w^{m-1} are conjugate to each other, (3) is also equivalent to

$$|A(w)|^2 + |B(w)|^2 = 2\left(m + \sum_{k=1}^{m-1} w^k\right). \tag{4}$$

Let p and q be respectively the numbers of -1's in each row of A and B. By replacing 1 by 0 and -1 by 1 in A and B and performing the similar process as above, we obtain the following formula corresponding to (4).

$$|C(w)|^2 + |D(w)|^2 = p + q + r\sum_{k=1}^{m-1} w^k, \tag{5}$$

TABLE I

m	$C(w)$	$D(w)$	N_A	N_B	N
3	0	1	1	1	1
5	1	1	1	1	1
7	1	$1 + w + w^3$	1	1	1
9	$1 + w$	$1 + w^2 + w^5$	3	3	3
13	$1 + w + w^3 + w^9$	$1 + w_0 + w_0^3 + w_0^9$	2	2	4
	$1 + w + w^4$	$1 + w + w^2 + w^4 + w^7 + w^9$ or $1 + w + w^3 + w^5 + w^7 + w^8$	2	4	4
15	$1 + w + w^4 + w^6$	$1 + w + w^3 + w^4 + w^8 + w^{10}$ or $1 + w + w^3 + w^5 + w^8 + w^9$	4	8	8
	$1 + w + w^4 + w^{10}$	$1 + w + w^3 + w^5 + w^7 + w^8$	2	2	2
19	$1 + w + w^3 + w^5 + w^8 + w^9$	$1 + w + w^3 + w^7 + w^9 + w^{10} + w^{14}$	9	9	9
	$1 + w + w^2 + w^4 + w^7 + w^{12}$ or $1 + w + w^3 + w^{12} + w^{14} + w^{15}$	$1 + w + w^3 + w^4 + w^8 + w^{10} + w^{14}$ or $1 + w + w^3 + w^5 + w^9 + w^{10} + w^{16}$	6	6	12
	$1 + w + w^2 + w^3 + w^6 + w^{12}$	$1 + w + w^5 + w^7 + w^9 + w^{12} + w^{15}$	9	9	9
	$1 + w + w^2 + w^3 + w^7 + w^{12}$	$1 + w + w^3 + w^6 + w^8 + w^{12} + w^{16}$ or $1 + w + w^3 + w^7 + w^{11} + w^{14} + w^{16}$	9	18	18

TABLE II

m	$C(w)$	$D(w)$	$N = N_A = N_B$
21	$1 + w + w^5 + w^7 + w^9 + w^{10}$	$1 + w + w^2 + w^3 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{17}$	6
23	$1 + w + w^2 + w^5 + w^7 + w^{11} + w^{14}$	$1 + w + w^2 + w^3 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{18}$	11
27	$1 + w + w^2 + w^6 + w^9 + w^{18} + w^{21} + w^{22}$	$1 + w + w^2 + w^3 + w^9 + w^{11} + w^{13} + w^{16} + w^{19} + w^{23} + w^{24}$	9

TABLE III

k	A corresponding to $C(w^k)$	B corresponding to $D(w^k)$
2	−+−+− +++−− ++−++ −+−−+ +++++ ++	−+−+− −−+++ +−+++ +++−− +−−++ +−
4	−++−− ++−−− +++++ +−+−+ ++++− ++	−+++− +++−− −−−++ −+−++ ++−++ −+
5	−+−−+ −+++− −++++ +++−+ −+++− ++	−−+++ −+−++ −−−+− −++−+ +++++ +−
7	−−+++ ++−+− ++−+− −++−− +++++ ++	−+++− +−−+− −+++− +++++ +−+−+ −−
8	−++++ −−+−− ++++− +−+−+ +−+++ ++	−++−+ ++−−+ +++++ +−−−+ −+−−− ++
10	−+++− +−++− −++−+ +++−+ −−+++ ++	−−−−+ ++++− −+++− +++++ −+−+− −+
11	−++++ ++++− +−−++ −+−−+ ++−++ +−	−++++ +−+−+ −−+−− +++−+ −−−++ ++
13	−−−−+ ++++− +−++− +++−+ ++++− ++	−+−+− ++−−− ++−−+ −+++− +++++ +−

where $C(w) = \sum_{k=0}^{m-1} c_k w^k$ with $c_k = 0$ whenever $a_k = 1$ and $c_k = 1$ whenever $a_k = -1$; $D(w)$ is similarly defined. The following relation is satisfied by r, when $w = 1$ is put into (5).

$$p^2 + q^2 = p + q + r(m-1)\,. \tag{6}$$

Similarly from (4), we have

$$(m-2p)^2 + (m-2q)^2 = 4m - 2\,. \tag{7}$$

When $w \neq 1$, from (5), (6), and (7), we obtain

$$|C(w)|^2 + |D(w)|^2 = \tfrac{1}{2}(m-1)\,. \tag{*}$$

All maximal $(+1, -1)$-matrices M_{2m} for odd m constructible by (1) with the restriction $p \leqq q < \frac{1}{2}m$, for $m \leqq 19$ were listed in [4].

Thus construction of maximal matrices is reduced to finding of polynomials $C(w)$, $D(w)$ satisfying (*). Table I is obtained according to this classification for $n = 2m \leqq 38$. In this table, N, N_A, N_B are respectively numbers of distinct types of matrices M_n, A, B; w and w_0 are any primitive mth roots of unity.

For example, when $n = 18$, $m = 9$; primitive 9th roots of unity are $w = \exp(2\pi i/9)$, w^2, w^4, w^k for $k = 5, 7, 8$. Since w^k and w^{m-k} are symmetric with respect to $w^m = 1$ which corresponds to the main diagonal, $C(w^k)$ and $C(w^{m-k})$ produce designs of the same type. Consequently we can omit the cases for w^k with $k \geqq 5$. $C(w) = 1 + w$ and $D(w) = 1 + w^2 + w^5$ produce the corresponding designs $--+++\ ++++$ and $-+-++\ -+++$ respectively. Similarly, $C(w^2) = 1 + w^2$ and $D(w^2) = 1 + w^4 + w$ produce the designs $-+-++\ ++++$ and $--++-\ ++++$ respectively. Likewise, $C(w^4) = 1 + w^4$ and $D(w^4) = 1 + w^8 + w^2$ produce $-+++-\ ++++$ and $-+-++\ +++-$. When $m = 19$, it is sufficient to consider primitive roots $w = \exp(2\pi i/19)$, w^k for $2 \leqq k \leqq 9$. For the case $C(w) = 1 + w + w^2 + w^4 + w^7 + w^{12}$; we have $C(w^2) = 1 + w^2 + w^4 + w^8 + w^{14} + w^5$, $C(w^3) = 1 + w^3 + w^6 + w^{12} + w^2 + w^{17} = w^{-2}C(w^2)$, and $C(w^5) = 1 + w^5 + w^{10} + w + w^{16} + w^3 = w^5 C(w^{-2})$, which produce the corresponding designs $-+-+-\ -++-+\ ++++-\ ++++,\ -+--+\ +-+++\ ++-++\ ++-+,$ and $--+-+\ -++++\ -++++\ +-++$ respectively. All of these three designs are of the same type and their finite sequences are equal to $-1, 3, 3, 3, 3, 3, -1, -1, 3$. Similarly it can be shown easily that $C(w^k)$ for $k = 4, 6, 9$ produces designs with the finite sequences $3, -1, -1, 3, -1, 3, 3, 3, 3$; for $k = 1, 7, 8$, it produces those with $3, 3, 3, -1, 3, -1, 3, 3, -1$. In general, it can be shown that designs produced by $C(w)$, $w^k C(w)$, and $w^h C(w^{-1})$, are of the same type for any integers k and h.

Table II is obtained by applying this method of finding polynomials $C(w)$ and $D(w)$ to the previously known designs for $m = 21, 23$, and 27.

For example, when $m = 27$, with primitive roots $w^k = \exp(2\pi k i/27)$, (Table III) designs of distinct types are obtained.

Addition. The following new designs for M_{50}, M_{54}, and M_{62} with the corresponding $C(w)$ and $D(w)$ have been found.

When $m = 25$, we have

$$\begin{aligned} C(w) &= 1 + w + w^2 + w^6 + w^8 + w^{10} + w^{11} + w^{14} + w^{15}\,, \\ D(w) &= C(w^7) = 1 + w^2 + w^5 + w^6 + w^7 + w^{14} + w^{17} + w^{20} + w^{23}\,. \end{aligned}$$

When $m = 27$, we have

$$C(w) = 1 + w + w^2 + w^3 + w^6 + w^{10} + w^{12} + w^{15} + w^{23},$$
$$D(w) = 1 + w + w^2 + w^5 + w^7 + w^9 + w^{10} + w^{17} + w^{20} + w^{21} + w^{23}.$$

When $m = 31$, we have

$$C(w) = 1 + w + w^2 + w^3 + w^4 + w^8 + w^{13} + w^{19} + w^{23} + w^{26},$$
$$D(w) = 1 + w + w^2 + w^3 + w^6 + w^7 + w^{10} + w^{12} + w^{14} + w^{15} + w^{17} + w^{18} + w^{24} + w^{26} + w^{28}.$$

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